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Abstract

We introduce a self-dual, noncommutative and noncocommutative Hopf algebra $\mathcal{H}_{GT}$ which takes for certain Hopf categories (and therefore braided monoidal bicategories) a similar role to the Grothendieck–Teichmüller group for quasitensor categories. We also give a result which highly restricts the possibility for similar structures for higher weak n-categories ($n \geqslant 3$) by showing that these structures would not allow for any nontrivial deformations. Finally, we give an explicit description of the elements of $\mathcal{H}_{GT}$.

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1. The Hopf algebra $\mathcal{H}_{GT}$

In [Dri] Drinfeld introduced the Grothendieck–Teichmüller group by considering the (formal) reparametrizations of the data (commutativity and associativity isomorphisms) of a quasitensor category. Consider now braided (weak) monoidal bicategories arising from the representations of a Hopf category (as defined in [CF]) on 2-vector spaces (see [KV]), i.e. on certain module categories. Let us assume, in addition, that the Hopf category itself is given as the category of finite-dimensional representations of a quasi-trialgebra, satisfying a quasitriangularity and coquasitriangularity condition. This is analogous to understanding the Grothendieck–Teichmüller group GT as a universal symmetry of quasitriangular quasi-Hopf algebras (see e.g. [CP]) which via their category of finite-dimensional representations then give rise to the aforementioned quasitensor categories. Considering the question of a universal symmetry of quasitriangular quasi-Hopf algebras is of interest, e.g., in the study of symmetries of moduli spaces of two-dimensional conformal field theories (see [Kon]) since two-dimensional conformal field theories are closely linked to the highest weight representation of quasitriangular quasi-Hopf algebras through their vertex algebras. Since the author has shown that any three-dimensional extended topological quantum field theory in the sense of [KL] uniquely determines a trialgebra ([Sch]) and these three-dimensional

extended topological quantum field theories are supposed to be related to two-dimensional boundary conformal field theories, the question of a universal symmetry of trialgebras is of potential interest to the question of symmetries on moduli spaces of two-dimensional boundary conformal field theories.

Remark 1. Note that the above-mentioned result, linking trialgebras to extended topological quantum field theories, also shows that the restriction of the consideration to Hopf categories which are representation categories of a trialgebra still includes a large and—from the physics perspective—the most important class of examples of such structures.

Let us begin by commenting on some of the involved notions or give references to the relevant literature, respectively. Especially, we will introduce the notion of a trialgebra in detail now. For the notion of (quasi-) Hopf algebras, quasitriangularity and coquasitriangularity etc, we refer to any of the many excellent introductions to Hopf algebras and quantum groups, available now (e.g. [CP] or [KS]). The notion of quasitensor category which is used in Drinfeld's definition of GT is given as a category together with a tensor product $\otimes$ on it where $\otimes$ need not be symmetric but satisfying a commutativity constraint 'up to isomorphism'. For the purpose of this paper, the reader should imagine a quasitensor category simply as the category of finite-dimensional representations of a quasitriangular quasi-Hopf algebra and the commutativity constraint to be given by a universal R-matrix. For a detailed introduction to quasitensor categories and their link to Hopf algebras and quantum groups, we refer to [CP].

Let us next introduce the concept of a trialgebra.

Definition 1. *A trialgebra $(A, *, \Delta, \cdot)$ with $*$ and $\cdot$ associative products on a vector space A (where $*$ may only be partially defined) and Δ a co-associative coproduct on A is given if both $(A, *, \Delta)$ and $(A, \cdot, \Delta)$ are bialgebras and the following compatibility condition between the products is satisfied for arbitrary elements $a, b, c, d \in A$,*

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

whenever both sides are defined.

Trialgebras were first suggested in [CF] as an algebraic means for the construction of four-dimensional topological quantum field theories. It was observed there that the representation categories of trialgebras have the structure of so-called Hopf algebra categories (see [CF]) and it was later shown explicitly in [CKS] that from the data of a Hopf category one can, indeed, construct a four-dimensional topological quantum field theory. The first explicit examples of trialgebras were constructed in [GS1] and [GS2] by applying deformation theory, once again, to the function algebra on the Manin plane and some of the classical examples of quantum algebras and function algebras on quantum groups. In [GS4] it was shown that one of the trialgebras constructed in this way appears as a symmetry of a two-dimensional spin system. Besides this, the same trialgebra can also be found as a symmetry of a certain system of infinitely many coupled q-deformed harmonic oscillators.

Definition 2. *We call a trialgebra quasitriangular (coquasitriangular) if one of the bialgebras contained in it is quasitriangular (coquasitriangular). We call a trialgebra $(A, \cdot, *, \Delta)$ biquasitriangular if $(A, \cdot, \Delta)$ is quasitriangular and $(A, *, \Delta)$ is coquasitriangular and if for the R-matrix $R_{\cdot}$ of $(A, \cdot, \Delta)$ and the linear form R_* expressing the coquasitriangularity of $(A, *, \Delta)$, the following condition holds,*

$$[R_{\cdot}, \widehat{R_*}] = 0$$

where $\widehat{R_*}$ is the R-matrix of a Hopf algebra dual to $(A, *, \Delta)$.

We will speak of a quasi-trialgebra if there is a Drinfeld co-associator α for Δ, and one of the two products has a dual associator β such that α and β satisfy a similar commutator condition as the R-matrices above (we will study this condition in detail in the next section).

Lemma 1. *The (formal) reparametrizations of the data of the above-mentioned biquasitriangular quasi-trialgebras define a self-dual noncommutative and noncocommutative Hopf algebra $\mathcal{H}_{GT}$.*

Proof. In a quasitriangular quasi-Hopf algebra, we have two kinds of data which are transformed by GT as a universal symmetry (see [Dri]): the R-matrix and the co-associator α. In the precise definition of GT, the completion of the transformations of these data with respect to a certain class of formal power series is considered (see [Dri] or [CP] for a comprehensive introduction to GT). In a biquasitriangular quasi-trialgebra, we have four types of data: the two matrices $R.$ and R_*, the co-associator α and the associator β. Now, we ask for the universal symmetry given by transformations of these data (including the same completion with respect to formal power series as in the case of GT).

First, observe that on the data $(R., \alpha)$ taken alone GT acts just by definition. Considering formal linear combinations of the data $(R., \alpha)$, we can, obviously, extend this to an action of the group algebra of GT (which naturally has the structure of a Hopf algebra, see e.g. [CP] or [KS]). Second, the class of all data (R_*, β) is dual to the class of all data $(R., \alpha)$. So, concerning a universal symmetry of the data (R_*, β) taken alone, we have to have a dual of the action of GT again. By the definition of the data (R_*, β), we cannot have an action of a group there but have to describe a universal symmetry by a co-action of a Hopf algebra. By the above argument, this has to be the function algebra on GT (where we define the appropriate function algebra as the algebra of polynomial functions, since GT is a projective limit of algebraic groups and the explicit definition of GT in [Dri] assures that the product of GT correctly transforms into a coproduct as one proves by calculation from the defining relations).

In consequence, if we would transform the data $(R., \alpha)$ and (R_*, β) of the two bialgebras included in a trialgebra separately, forgetting about the compatibility condition for the two products of a trialgebra, the universal symmetry would be described by the Drinfeld double $\mathcal{D}(GT)$ of GT, i.e. the tensor product of the group algebra of GT and the algebra of functions on GT. In the next step, we have to restrict to those transformations of the complete set of data $R., R_*, \alpha, \beta$ which transform a biquasitriangular quasi-trialgebra into a biquasitriangular quasi-trialgebra. Obviously, this is a subspace of $\mathcal{D}(GT)$. One proves by calculation from the compatibility relation of the two products that it is a sub-Hopf algebra $\mathcal{H}_{GT}$ indeed.

It remains to show that $\mathcal{H}_{GT}$ is self-dual, noncommutative and noncocommutative: the self-duality follows from the fact that the classes of data $(R., \alpha)$ and (R_*, β) are dual to each other. $\mathcal{H}_{GT}$ cannot be commutative since one of the factors of $\mathcal{D}(GT)$ restricts to the group algebra of GT and GT is non-Abelian. Finally, $\mathcal{H}_{GT}$ is noncocommutative, then, since it is self-dual. This completes the proof. $\qquad\square$

Lemma 2. *There is an algebra morphism from $\mathcal{H}_{GT}$ to the group algebra of the Grothendieck–Teichmüller group.*

Proof. Since, as shown above, $\mathcal{H}_{GT}$ is a sub-Hopf algebra of $\mathcal{D}(GT)$ and one of the factors of $\mathcal{D}(GT)$ is just the group algebra of GT, the conclusion follows. $\qquad\square$

Remark 2. Observe that the above map is only a morphism with respect to the associative algebra structure of $\mathcal{H}_{GT}$. Also, it cannot be surjective since there are compatibility constraints between the commutativity and associativity isomorphisms and their dual structures.

One could have the idea to extend this approach to higher braided weak monoidal weak n-categories beyond the level of bicategories where for tricategories one would expect an algebraic structure in the form of a vector space equipped with two associative products and two co-associative coproducts to generate these tricategories via representation theory. We will call such an algebraic structure a quadraalgebra. The compatibilities are given by requiring that any of the coproducts together with the two products defines a trialgebra plus the requirement that the two coproducts are compatible by the dual relation to the compatibility relation for the two products.

Lemma 3. *There do not exist nontrivial deformations of a trialgebra into a quadraalgebra.*

Proof. With similar arguments as given above, one can show that the universal symmetry of a quasi-quadraalgebra, with two quasitriangularity conditions and one coquasitriangularity condition satisfied, is given by a trialgebra $\mathcal{T}_{GT}$ where there is an algebra morphism from $\mathcal{T}_{GT}$—as an associative algebra—to the group algebra of GT. Besides this, one proves that both bialgebras (which are Hopf algebras, even, in this case) within $\mathcal{T}_{GT}$ are self-dual and the two products have to agree (this follows again from the symmetry of the classes of data on which $\mathcal{T}_{GT}$ acts as a universal symmetry). Both products are, as a consequence, universally defined then. Since both products agree, we can without loss of generality assume that we have a unital product. But then we can apply an Eckmann–Hilton-type argument to conclude that the product is Abelian. So, $\mathcal{T}_{GT}$ is a commutative and cocommutative self-dual Hopf algebra. Besides this, $\mathcal{T}_{GT}$ is a sub-Hopf algebra of $\mathcal{D}(GT)$. But because of the algebra morphism from $\mathcal{T}_{GT}$ to the group algebra of GT, given by the previous lemma, $\mathcal{T}_{GT}$ is determined by an Abelian subgroup of GT then. But by definition of GT (see [Dri]), we get triviality then, i.e. $\mathcal{T}_{GT}$ consists—up to rational factors—of the identity only. But triviality of $\mathcal{T}_{GT}$ means that we cannot have a nontrivial formal deformation theory of quasi-quadraalgebras with suitable (co)quasitriangularity conditions. Remembering that in the definition of GT the co-associator is the essential part of the data (see [Dri] where this is already noted), we can extend this conclusion to general quasi-quadraalgebras. This concludes the proof.   □

So, on the level of tricategories arising via representation theory from quadraalgebras (and, consequently, for higher categorical levels linked to corresponding higher algebras), one gets only generalizations of braided monoidal structures which do not allow for deformations. Especially, as we have just shown, there is no nontrivial deformation theory of trialgebras into quadraalgebras, further generalizing the deformation of groups into Hopf algebras into trialgebras. So, on the level of trialgebras a kind of stability is reached. Observe that this nonexistence of deformations is much stronger than the usual rigidity in cohomology theory since we can not only exclude deformations in a given category of structures but also deformations to higher categorical analogues of the structure. For example, the usual rigidity results in the theory of classical Lie algebras do not exclude the deformation of the universal envelope into a noncommutative and noncocommutative Hopf algebra but—as we just mentioned—we can exclude deformations of trialgebras into algebraic structures involving four or more products and coproducts joined in a compatible way. We suggest the term *ultrarigidity* for this kind of stability.

Remark 3. Since Hopf categories are linked to four-dimensional topological field theory (as Hopf algebras are to the three-dimensional case), see [CF] and [CKS], this seems on the algebraic level to mirror the fact that geometry in dimension five and higher is in some sense much simpler than the three- and four-dimensional cases. There is also a more physical interpretation of this result: since bialgebra categories are linked supposedly to certain types

of quantum field theories on noncommutative spaces (see [GS3]), we can say that quantum field theory on noncommutative spaces is a stable structure in some sense, not allowing for a further generalization of the passage from classical to quantum field theory on noncommutative spaces.

So far, we have seen only a few abstract properties of the Hopf algebra $\mathcal{H}_{GT}$. We will give a more explicit description of $\mathcal{H}_{GT}$ in the next section.

2. The explicit structure of $\mathcal{H}_{GT}$

In [Dri] an explicit description of the Grothendieck–Teichmüller group GT is derived from the general definition of the group of transformations of the associator and the braiding of a quasitensor category. In this section, we want to do the same for the Hopf algebra $\mathcal{H}_{GT}$.

Recall that the elements of GT can be written in the form (λ, f) with $\lambda \in \mathbb{Q}$ and f belongs to the $\mathbb{Q}$-pro-unipotent completion of the free group of two generators where the pairs (λ, f) satisfy certain conditions (see [Dri]). Remember also that f arises in the following way from the general definition of GT. If we change the associator

$$(U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

this means multiplying it by an automorphism of $(U \otimes V) \otimes W$. It can be shown that any such automorphism is of the form

$$f\left(\sigma_1^2, \sigma_2^2\right)(\sigma_1 \sigma_2)^{3n}$$

with $n \in \mathbb{Z}$ and f as above. Here, σ_1, σ_2 are the generators of the braid group B_3.

Now, assume that we have a Hopf category (see [CF]) with associativity isomorphism α for the tensor product and co-associativity isomorphism β for the functorial coproduct. Note that while the possible transformations of α are represented by automorphisms of tensor products $(U \otimes V) \otimes W$, the possible cotransformations of β are of a dual nature and can formally be seen as elements of the algebraic dual of the underlying vector space of the automorphism group

$$\text{Aut}((U \otimes V) \otimes W)$$

(remember that a Hopf category is, especially, $\mathbb{C}$-linear). So, cotransformations of β can be—up to linear combinations—written in the form $\widehat{g}$ where g is the second component of an element of GT and $\widehat{}$ denotes the dualization operation as defined above.

Next, remember that the structure of GT is basically determined by the transformations of the associator (see [Dri, Kon]), i.e. in what follows, we will forget about the component λ coming from the braiding.

In conclusion, we can describe the Hopf algebra $\mathcal{H}_{GT}$ as a sub-Hopf algebra of the tensor product of the function algebra of GT with the Hopf algebra dual of GT (as defined above), i.e. as a sub-Hopf algebra of the Drinfeld double of GT.

In order to determine the concrete nature of this sub-Hopf algebra, we have to use the compatibility condition between α and β involved in the definition of a Hopf category. While for objects U, V, W α is represented as an isomorphism

$$\varphi : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

β is, again, given by an element $\widehat{\psi}$ of the dual of the space of such transformations. The natural compatibility condition is, then,

$$\psi^{-1}\varphi = \varphi^{-1}\psi. \tag{1}$$

In order to assure that a pair $(f, \widehat{g})$ transforms a Hopf category into a Hopf category, we require that for the transformed isomorphisms also the above equation holds. Since f and g act on φ and ψ, respectively, by the multiplication

$$\varphi \mapsto \varphi f \qquad \psi \mapsto \psi g$$

it follows that

$$g^{-1}\psi^{-1}\varphi f = f^{-1}\varphi^{-1}\psi g. \tag{2}$$

Let

$$\chi = \psi^{-1}\varphi$$

i.e. equation (2) reads

$$g^{-1}\chi f = f^{-1}\chi^{-1}g. \tag{3}$$

We are searching for a universal structure (i.e. not dependent on the choice of Hopf category) of $\mathcal{H}_{GT}$, so, we require that (3) holds for all possible choices of χ.

Since equation (1) implies that

$$\chi^2 = 1$$

i.e. χ is a projector, it follows that equation (3) holds for all possible choices of χ iff it holds for χ being the identity. So, equation (2) is equivalent to the condition

$$g^{-1}f = f^{-1}g \tag{4}$$

for the elements f, g of GT, i.e. we have a universal condition determining $\mathcal{H}_{GT}$.

Remark 4. The symmetry inherent in condition (4) is, of course, the source of the self-duality of $\mathcal{H}_{GT}$.

Remark 5. Mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$ are believed to be given as representations of GT (see [Kon]). The explicit nature of condition (4), in principle, allows for explicit calculations of the quantum analogues of such motives as representations of $\mathcal{H}_{GT}$ which are also corepresentations of $\mathcal{H}_{GT}$. Given pairs of representations of GT, one can use (4) to determine such representations of $\mathcal{H}_{GT}$. On the other hand, the condition also shows that one has to expect the representations of pairs $(f, \widehat{g})$ satisfying (4) to exist where neither of the components f or g derives from a full representation of GT, i.e. one has to expect quantum motives which do not derive from a classical counterpart. For example, partial representations of GT which would develop singularities, if one tries to extend them to a full representation, could play a role here.

We want to conclude this section with another small observation. In [KL] an algebraic framework—so called extended topological quantum field theories—is developed in detail which allows for the inclusion of the case of boundary conformal field theories into the algebraic description. It is shown there that such theories are determined by modular categories $\mathcal{C}$ (i.e. certain quasitensor categories) together with a Hopf algebra object H in $\mathcal{C}$. One can immediately define a category $\mathrm{Rep}(H)$ of representations of H in $\mathcal{C}$ from this.

Lemma 4. *The possible compatible transformations of* $\mathrm{Rep}(H)$ *together with* $\mathcal{C}$ *are determined by pairs* (f, g) *of elements* f, g *of* GT *satisfying condition (4).*

Proof. Direct consequence of the definition of $\mathrm{Rep}(H)$. □

So, from the view of the abstract quantum symmetry $\mathcal{H}_{GT}$, the algebraic formulation of boundary conformal field theories given by [KL] and the structure of trialgebras and Hopf categories are just different concrete realizations of one and the same quantum symmetry.

Remark 6. One can dually also formulate $\mathcal{H}_{GT}$ by starting from the Ihara algebra Ih (see [Dri, Iha1, Iha2] for the definition) instead of GT (the Ihara algebra is closely related to the Lie algebra of GT). Condition (4) translates then to the condition

$$[f, h] = 0 \tag{5}$$

for elements $f, h \in$ Ih.

Remembering that the Lie algebra structure of Ih derives—by evaluation of the elements of Ih on finite-dimensional metrized (i.e. endowed with an invariant inner product) Lie algebra g—from the Kirillov bracket (see [Dri] and for the definition of the Kirillov bracket [Kir]), condition (5) translates after evaluation on g to

$$\{f_g, h_g\} = 0$$

i.e. we can view it as requiring h_g to behave as a symmetry relative to f_g and vice versa.

3. Conclusion

We have introduced a noncommutative analogue $\mathcal{H}_{GT}$ of the Grothendieck–Teichmüller group in the form of a self-dual, noncommutative and noncocommutative Hopf algebra. Besides this, we have given an explicit description of the elements of $\mathcal{H}_{GT}$. We also proved a stability property (*ultrarigidity*) excluding deformations of certain higher categorical structures than bicategories. Further work will deal, in particular, with physical applications of this stability result.

Acknowledgments

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